CORINGS IN THE CATEGORY OF RINGS

By

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To my mother

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A coring in a category $\mathcal{C}$ is an object A of $\mathcal{C}$ together with morphisms $a,m:A \longrightarrow A*A$ inducing binary operations on the morphism set $[A,X]$ such that with these induced operations, $[A,X]$ becomes a ring, denoted by $[A,X]_{a,m}$, for any object X in $\mathcal{C}$. In Chapter I, it is shown that many familiar functors on $\mathcal{R}$, the category of rings, can be represented as $[A,-]_{a,m}$ where (A,a,m) is a coring in $\mathcal{R}$.

The second chapter presents several examples of functors on the category of rings which have left adjoints. It also proves the result, a special case of a theorem of Freyd, that a functor on $\mathcal{R}$ has a left adjoint if and only if it is representable as $[A,-]_{a,m}$ for some coring (A,a,m) in $\mathcal{R}$.

A coring (C,a,m) in $\mathcal{R}$ is said to be standard if C is the free ring on a set X and $a(x) = u_1(x) + u_2(x)$ for each x in X where u_1,u_2 are the injections into the coproduct $C*C$. Chapter III deals with the types of functors representable

by standard corings and characterizes those coring functors which can be expressed as $R \otimes -$ for some ring R.

The fourth chapter discusses the category of corings in $\mathfrak{R}$ and its properties.

The last chapter is concerned with cosemigroups in the category of semigroups. It is shown that the category of semigroups has exactly two auto-equivalences, the identity functor and the opposite functor.

INTRODUCTION

A binary operation on an object A of a category is a morphism $m:A\times A \longrightarrow A$. Dually, if $\mathcal{C}$ is a category with finite coproducts, then a binary co-operation on an object A is a morphism $m:A \longrightarrow A*A$, from A to the coproduct of A with itself. A binary co-operation m induces a binary operation on the set of morphisms $[A,X]$ for any object X in the category. The co-operation is said to be associative (commutative, etc.) if the induced operation on $[A,X]$ has the property for every X in $\mathcal{C}$. A coring in $\mathcal{C}$ is an object A together with co-operations a,m on A such that with the operations induced by a and m, $[A,X]$ is a ring for any object X in $\mathcal{C}$. If we denote this ring by $[A,X]_{a,m}$, then $[A,-]_{a,m}$ becomes a functor from $\mathcal{C}$ to $\mathcal{R}$, the category of rings.

Many functors on $\mathcal{R}$, for example, the functor which sends a ring to its n x n matrix ring, can be represented as $[A,-]_{a,m}$ for an appropriate coring (A,a,m) in $\mathcal{R}$. Also, the functors on $\mathcal{R}$ which have left adjoints are precisely those which are representable by coring functors.

This dissertation deals with several aspects of corings

and functors with left adjoints on the category of rings. The first two chapters show examples of corings and adjoint functors on $\mathfrak{R}$ and prove the fact stated above--that functors with left adjoints are the representable functors. Chapter III deals with the structure of corings in $\mathfrak{R}$ and shows the relationship between coring functors and functors which can be expressed as $R \otimes -$ for some ring R. The fourth chapter is concerned with the category of corings and its properties. The last chapter looks at cosemigroups in the category of semigroups and proves the result, parallel to a result of Clark for the category of rings, that the category of semigroups has exactly two automorphisms, the identity functor and the opposite functor.

CHAPTER I

CORINGS

A binary operation on a set A is a mapping from $A \times A \longrightarrow A$. If $\mathcal{C}$ is a category with finite products, then a binary operation on an object A in the category would be a morphism $m: A \times A \longrightarrow A$. Dually, if $\mathcal{C}$ is a category with finite coproducts, then a <u>binary co-operation</u> on an object A is a morphism $m: A \longrightarrow A*A$, from A to the coproduct of A with itself.

If $m: A \longrightarrow A*A$ is a co-operation in the category $\mathcal{C}$, and if X is any object in $\mathcal{C}$, then m induces a binary operation on the set of morphisms $[A,X]$ in the following manner: Suppose $f,g: A \longrightarrow X$. Let $u_1, u_2: A \longrightarrow A*A$ be the injections into the coproduct, and denote by $\langle f,g \rangle$ the unique morphism from $A*A$ to X such that $\langle f,g \rangle u_1 = f$ and $\langle f,g \rangle u_2 = g$. Then define $f \cdot g = \langle f,g \rangle m$. The co-operation m is said to be associative (commutative, have an identity, etc.) if the induced operation on $[A,X]$ has this property for every object X in $\mathcal{C}$. If $a,m: A \longrightarrow A*A$ are co-operations inducing operations $+, \cdot$ on $[A,X]$ for all X in $\mathcal{C}$, and if $([A,X],+,\cdot)$ is a ring for all X in $\mathcal{C}$, then (A,a,m) is called a <u>coring</u> in the category $\mathcal{C}$. Analagously, one could define a <u>cogroup</u>, <u>cosemigroup</u>, <u>co-monoid</u>, etc. We will denote the ring $([A,X],+,\cdot)$ by

$[A,X]_{a,m}$.

If (A,a,m) is a coring, and if $\alpha:X\longrightarrow Y$ is a morphism in $\mathcal{C}$, then define $[A,\alpha]:[A,X]_{a,m}\longrightarrow[A,Y]_{a,m}$ by $[A,\alpha](f) = \alpha f$ for f in $[A,X]$.

(1.1) Lemma: Let (A,a,m) be a coring in $\mathcal{C}$. Then $[A,-]:\mathcal{C}\longrightarrow\mathcal{R}$ is a functor, where $\mathcal{R}$ denotes the category of rings.

Proof: Suppose $\alpha:X\longrightarrow Y$ is an $\mathcal{C}$-morphism. We must show that $[A,\alpha]$ is actually a ring homomorphism. Suppose $f,g \in [A,X]_{a,m}$.

$$[A,\alpha](f + g) = [A,\alpha](\langle f,g\rangle a) = \alpha\langle f,g\rangle a = \langle\alpha f,\alpha g\rangle a = \alpha f + \alpha g = [A,\alpha](f) + [A,\alpha](g).$$

A similar argument holds for multiplication, so $[A,\alpha]$ is indeed a ring homomorphism.

If $1_X:X\longrightarrow X$ is the identity morphism, then $[A,1_X](f) = 1_X f = f$ for any f in $[A,X]_{a,m}$, so $[A,1_X]_{a,m}$ is the identity map on $[A,X]_{a,m}$. If $\alpha:X\longrightarrow Y$ and $\beta:Y\longrightarrow Z$ are $\mathcal{C}$-morphisms, and if $f \in [A,X]_{a,m}$, then $[A,\beta\alpha](f) = \beta\alpha f = [A,\beta][A,\alpha](f)$. Thus, $[A,\beta\alpha] = [A,\beta][A,\alpha]$, and $[A,-]$ is a functor.

A functor $T:\mathcal{C}\longrightarrow\mathcal{R}$ is said to be representable if it is naturally equivalent to $[A,-]_{a,m}$ for some coring (A,a,m) in $\mathcal{C}$.

Let S be a semigroup with 0. If R is a ring, then

define the ring $R_0[S]$ as follows:

$R_0[S] = \{\sum_{s\neq 0} r_s s \mid r_s \in R$ and only a finite number of the r_s's are nonzero$\}$.

$(\Sigma r_s s) + (\Sigma t_s s) = \Sigma(r_s + t_s)s$

$(\Sigma r_s s)\cdot(\Sigma t_s s) = \Sigma u_s s$, where $u_s = \sum_{vw=s} r_v t_w$.

$R_0[S]$ is called the <u>contracted semigroup ring of S over R</u>. If S is any semigroup with 0 with the property that every nonzero element has only a finite number of factors, then the ring $R_0[[S]]$ can be defined just like $R_0[S]$, except that the restriction that finitely many of the r_s's be zero is removed. $R_0[[S]]$ is called the <u>contracted power series semigroup ring of S over R</u>. If S is finite, then $R_0[[S]]$ and $R_0[S]$ coincide.

If S is the semigroup of n x n matrix units, then $R_0[S]$ is R_n, the ring of n x n matrices over R. If $S = \{0,1,x,x^2,\ldots\}$, the infinite cyclic semigroup with identity and zero, then $R_0[[S]]$ is the ring of formal power series over R, while $R_0[S]$ is the ring of polynomials in one indeterminate over R.

Let $\mathfrak{R}$ be the category of rings, and let S be a semigroup with zero such that every nonzero element has only a finite number of factors. Then there is a functor T_S from $\mathfrak{R}$ to $\mathfrak{R}$ defined by $T_s(R) = R_0[[S]]$, and if $f:R\longrightarrow R'$, then

$T_S(f):T_S(R) \longrightarrow T_S(R')$ by $T_S(f)(\Sigma r_s s) = \Sigma f(r_s)s$. If S is any semigroup, then there is a functor F_S such that $F_S(R) = R_0[S]$ and $F_S(f)(\Sigma r_s s) = \Sigma f(r_s)s$. Note that if S is finite, then $T_S = F_S$.

(1.2) A functor of the type T_S can always be represented as $[A,-]_{a,m}$ for an appropriate coring (A,a,m). Let $A = W(S - \{0\})$, the free ring on the set of nonzero elements of S. To define a and m, it is sufficient to define them only on the generators of A, the nonzero elements of S. Define:

$$a(s) = u_1(s) + u_2(s)$$

$$m(s) = \sum_{rt=s} u_1(r)u_2(s)$$

$$= 0 \text{ if } s \text{ has no factors.}$$

To show that T_S and $[A,-]_{a,m}$ are naturally equivalent, we define the following natural equivalence: If B is a ring, define $\eta_B:[A,B]_{a,m} \longrightarrow B_0[[S]]$ by $\eta_B(f) = \sum_{s\in S-\{0\}} f(s)s$. We must show that η_B is a ring homomorphism.

$$\eta_B(f + g) = \eta_B(\langle f,g\rangle a) = \Sigma\langle f,g\rangle a(s)s$$

$$= \Sigma\langle f,g\rangle(u_1(s) + u_2(s))s = \Sigma(f(s) + g(s))s$$

$$= \Sigma f(s)s + \Sigma g(s)s = \eta_B(f) + \eta_B(g).$$

$$\eta_B(f\cdot g) = \eta_B(\langle f,g\rangle m) = \Sigma\langle f,g\rangle m(s)s$$

$$= \sum_{s\in S-\{0\}} \langle f,g\rangle \left(\sum_{rt=s} u_1(r)u_2(t)\right)s$$

$$= \sum_{s \in S-\{0\}} \left(\sum_{rt=s} \langle f,g\rangle u_1(r)u_2(t) \right) s$$

$$= \sum_{s \in S-\{0\}} \left(\sum_{rt=s} f(r)g(t) \right) s$$

$$= (\Sigma f(s)s)\ (\Sigma g(s)s).$$

So η_B is a ring homomorphism.

η_B is one-to-one, since if $\eta_B(f) = \eta_B(g)$, then $\Sigma f(s)s = \Sigma g(s)s$; $f(s) = g(s)$ for all s, and thus $f = g$.

η_B is onto, for if $\Sigma b_s s \in B_0[[S]]$, there is a ring homomorphism $f:A \longrightarrow B$ such that $f(s) = b_s$ for all s. Then $\eta_B(f) = \Sigma b_s s$.

Thus, η_B is an isomorphism for each B. Up until this point, we didn't really know $[A,B]_{a,m}$ is a ring, but since $[A,B]_{a,m}$ has two operations on it and with these operations is isomorphic to a ring, it must also be a ring, and so (A,a,m) is a coring.

To see that η is natural, we must show that the following diagram is commutative for all $\alpha:B \longrightarrow C$

$$\begin{tikzcd}
{[A,B]_{a,m}} \arrow[r, "\eta_B"] \arrow[d, "{[A,\alpha]}"] & T_S(B) \arrow[d, "T_S(\alpha)"] \\
{[A,C]_{a,m}} \arrow[r, "\eta_C"] & T_S(C)
\end{tikzcd}$$

But if $f \in [A,B]_{a,m}$, then

$$T_S(\alpha)\,\eta_B(f) = T_S(\alpha)(\Sigma f(s)s) = \Sigma\alpha f(s)s;$$

$$\eta_C[A,\alpha]_{a,m}(f) = \eta_C(\alpha f) = \Sigma\alpha f(s)s.$$

Thus, the two functors are naturally equivalent.

We will show later (2.9) that if S is an infinite semigroup, then a functor of the type F_S cannot be representable.

If S is the semigroup $\{0,1\}$ under multiplication, then it is easy to see that $R_0[S] \cong R$ for any ring R, and that T_S is thus naturally equivalent to the identity functor. Thus, the identity functor on $\mathfrak{R}$ is representable, and, using the construction for the coring described above, we get:

(1.3) <u>Proposition</u>: Let $G = W(\{x\})$, the free ring on one generator, and let a, m be defined by:

$$a(x) = u_1(x) + u_2(x)$$

$$m(x) = u_1(x)u_2(x).$$

Then (G,a,m) is a coring and $[G,-]_{a,m}$ is naturally equivalent to the identity functor on $\mathfrak{R}$.

Functors on the category of rings other than semigroup functors can be represented in a similar manner. For example, let T be the functor which takes a ring R to the additive group $R \oplus R$ with multiplication $(r,s)(r',s') = (rr' - ss', sr' + rs')$ (complex-number-like multiplication).

Then T can be represented as $[C,-]_{a,m}$ where $C = W(x_1,x_2)$, the free ring on two generators, and coaddition and comultiplication are given by:

$$a(x_i) = u_1(x_i) + u_2(x_i),\ i = 1,2.$$

$$m(x_1) = u_1(x_1)u_2(x_1) - u_1(x_2)u_2(x_2)$$

$$m(x_2) = u_1(x_1)u_2(x_2) + u_1(x_2)u_2(x_1).$$

The functor which sends a ring to its ring of quaternions can also be represented in a similar manner. This will be generalized later.

It would be of interest to characterize categorically some of the more common functors on the category of rings, for example, the functor which takes a ring R into R_n. This functor is of the form T_S (or F_S) where S is the semigroup $\{e_{ij}\} \cup \{0\}$ of matrix units. The next proposition shows that this functor is cokernel preserving.

(1.4) Proposition: Let S be a semigroup with 0. Then F_S is cokernel preserving if and only if $S^2 = S$.

Proof: If $f:R \longrightarrow R'$ is a ring homomorphism, then coker(f) can be realized as R'/I, where $I = \langle im(f)\rangle_{id}$, the ideal of R' generated by the image of f.

Suppose F_S is cokernel preserving, and suppose x is in $S - S^2$. Consider $i:Z \longrightarrow Q$, the inclusion map from the

integers into the rational numbers. Coker(i) = 0, so coker $(T_S(i)) = T_S(0) = 0$. If I is the ideal of $Q_0[S]$ generated by $\mathrm{im}(T_S(i))$, then $Q_0[S]/I = 0$, so $Q_0[S] = I$. $\frac{1}{2}x \in I$ (otherwise x would have to be in S^2), and we have a contradiction. Thus $S = S^2$.

Conversely, suppose $S = S^2$. T_S is cokernel preserving if $R'[S]/\langle \mathrm{im}(T_S(f)\rangle_{id} \cong (R'/\langle \mathrm{im}(f)\rangle_{id})[S]$ for any $f:R \longrightarrow R'$. Since $(R'/\langle \mathrm{im}(f)\rangle_{id})[S] \cong R'[S]/\langle \mathrm{im}(f)\rangle_{id}[S]$, it is sufficient to show $\langle \mathrm{im}(T(f))\rangle_{id} = (\langle \mathrm{im}(f)\rangle_{id})[S]$. Suppose $x \in \langle \mathrm{im}(T(f))\rangle_{id}$. Then

$$x = T(f)(\Sigma r_i s_i) + (\Sigma t_i' s_i)T(f)(\Sigma t_i s_i) +$$

$$+ T(f)(\Sigma u_i s_i)\Sigma u_i' s_i) + (\Sigma r_i' s_i)T(f)(\Sigma v_i s_i)(\Sigma v_i' s_i),$$

where each $r_i, t_i, u_i, v_i \in R$, and $r_i', t_i', u_i', v_i' \in R'$. Then

$$x = \sum_{i=1}^{n}(\sum_{s_j s_k = s_i} f(u_j)u_k')s_i + \sum_{i=1}^{n}(\sum_{s_j s_k s_l = s_i} r_j' f(v_k)v_l')s_i.$$

Noting that each coefficient is in $\langle \mathrm{im}(f)\rangle_{id}$, we see that $x \in \langle \mathrm{im}(f)\rangle_{id}[S]$.

Now suppose $x \in (\langle \mathrm{im}(f)\rangle_{id})[S]$. Then $x = \sum_{i=1}^{n} w_i s_i$ where $w_i \in (\langle \mathrm{im}(f)\rangle_{id})$. Then

$$w_i = f(a_i) + b_i' f(b_i) + f(c_i)c_i' + a_i' f(d_i)d_i'$$

where $a_i, b_i, c_i, d_i \in R$ and $a_i', b_i', c_i', d_i' \in R'$. Since $S^2 = S$, $s_i = t_i u_i$ for each i, and $u_i = v_i z_i$.

$$\begin{aligned} x &= \sum_{i=1}^{n} f(a_i)s_i + \sum_{i=1}^{n} b_i' f(b_i)s_i + \sum_{i=1}^{n} f(c_i)c_i' + \sum_{i=1}^{n} a_i' f(d_i)d_i' \\ &= \sum_{i=1}^{n} f(a_i)s_i + \sum_{i=1}^{n} (b_i't_i)(f(b_i)u_i) + \\ &\quad + \sum_{i=1}^{n} (f(c_i)t_i)(c_i'u_i) + \sum_{i=1}^{n} (a_i't_i)f(d_i)v_id_i'z_i \\ &= \sum_{i=1}^{n} T(f)(a_is_i) + \sum_{i=1}^{n} (b_i't_i)(T(f)(b_iu_i)) + \\ &\quad + \sum_{i=1}^{n} T(f)(c_it_i)c_i'u_i + \sum_{i=1}^{n} (a_i't_i)T(f)(d_iv_i)(d_i'z_i). \end{aligned}$$

But each term of each summation is in $\langle \mathrm{im}(T(f))\rangle_{id}$, so $x \in \langle \mathrm{im}(T(f))\rangle_{id}$.

The functor $R \longrightarrow R_n$ has the property that it takes simple rings into simple rings. It follows easily from the next proposition that T_S cannot have this property if S is infinite.

(1.5) <u>Proposition</u>: Let S be an infinite nontrivial semigroup with 0 with the property that any nonzero element has at most a finite number of factors. Then S is not simple.

<u>Proof</u>: Suppose S is simple. Since S is nontrivial, $S^2 \neq \{0\}$. S^2 is an ideal, so $S^2 = S$, and $SSS = S$. For any $x \neq 0$, consider $T = \{s \in S | SsS = \{0\}\}$. T is clearly an ideal of S. $T \neq S$, for then $SSS = \{0\}$, so $T = 0$. Thus $SxS \neq \{0\}$ for any $x \neq 0$, and SxS is an ideal, so $SxS = S$ for any $x \neq 0$. Now consider y in S. $y \in SxS$ for all $x \neq 0$, so for each x,

there exist s_x, t_x such that $y = s_x(xt_x)$. But y has only a finite number of factors, so only a finite number of these factorizations can be distinct. But then there must be x such that $xt_x = yt_y$ for infinitely many y, a contradiction.

<u>Corollary</u>: If S is as in the above proposition, T_S does not take simple rings into simple rings.

<u>Proof</u>: If I is an ideal of S, then for any ring R, $R_0[[I]]$ is an ideal of $R_0[[S]]$.

CHAPTER II

ADJOINT FUNCTORS

One of the most fruitful areas of study in category theory has been the theory of adjoint functors. In this section we will investigate those functors on the category of rings which have left adjoints.

If $T:\mathcal{A} \longrightarrow \mathcal{B}$ and $S:\mathcal{B} \longrightarrow \mathcal{A}$ are covariant functors, then S is the left adjoint for T if there exists a natural equivalence of set-valued bifunctors

$$\eta_{B,A}:[S(B),A] \longrightarrow [B,T(A)].$$

We will make use of the following characterization:

(2.1) <u>Theorem</u>: A functor $T:\mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint if and only if for any object B in $\mathcal{B}$ there exists $S(B)$ in $\mathcal{A}$ and $v_B:B \longrightarrow TS(B)$ such that for any $\varphi:B \longrightarrow T(A)$, there exists a unique $\varphi':S(B) \longrightarrow A$ such that

$$\begin{tikzcd} B \arrow[r, "v_B"] \arrow[dr, "\varphi"'] & TS(B) \arrow[d, "T(\varphi')"] \\ & T(A) \end{tikzcd}$$

is commutative.

For a proof of this result, the reader is referred to [1, p. 119].

The functors on the category of rings which have left adjoints are precisely those which are representable. In fact, Freyd [2] has stated the following more general result:

(2.2) Let $\mathcal{C}$ be a complete category, $\mathcal{V}$ an algebraic variety, and $T:\mathcal{C}\longrightarrow\mathcal{V}$ a covariant functor. T is representable if and only if T has a left adjoint.

A proof discovered independently of Freyd is included here for the special case of a functor from the category of rings to itself. The proof is in two parts.

(2.3) Theorem: Suppose $T:\mathcal{R}\longrightarrow\mathcal{R}$ has a left adjoint S. If (C,a,m) is a coring in $\mathcal{R}$, then $(S(C),S(a),S(m))$ is also a coring in $\mathcal{R}$. In fact, $[S(C),-]_{S(a),S(m)} \cong [C,T(-)]_{a,m}$.

Proof: Since S has a right adjoint, S preserves co-products, so $S(a),S(m):S(C)\longrightarrow S(C*C)$ are actually co-operations. From the adjoint situation there is a natural equivalence $\eta:[S(C),-]\longrightarrow[C,T(-)]$ of set valued functors. We will show that for any ring R, η_R^{-1} is a ring homomorphism. It is known from the theory of adjoint functors that there is a natural transformation $\Psi:ST\longrightarrow 1_{\mathcal{R}}$ such that if $f:C\longrightarrow T(R)$, then $\eta_R^{-1}(f) = \Psi_R\, S(f)$. Now suppose $f,g \in [C,T(R)]$. Then

$$\eta_R^{-1}(f + g) = \eta_R^{-1}(\langle f,g\rangle a) = \Psi_R S(\langle f,g\rangle a) = \Psi_R\langle S(f),S(g)\rangle S(a)$$

$$\langle\eta_R^{-1}(f),\eta_R^{-1}(g)\rangle S(a) = \eta_R^{-1}(f) + \eta_R^{-1}(g)$$

and similarly for multiplication. Thus, η_R^{-1} is a ring isomorphism, $[S(C),R]_{S(a),S(m)}$ is a ring for any R, and $(S(C),S(a),S(m))$ is a coring. $\eta:[S(C),-]\longrightarrow[C,T(-)]$ is a natural equivalence in $\mathcal{R}$.

Corollary: If $T:\mathfrak{R}\longrightarrow\mathfrak{R}$ has a left adjoint S, then T is representable.

Proof: Let (G,a,m) be the coring described in Proposition (1.3) representing the identity functor. Then $[S(G),-]_{S(a),S(m)} \cong [G,T(-)]_{a,m} \cong T$.

(2.4) Theorem: Let (C,a,m) be a coring in $\mathfrak{R}$. Then $[C,-]_{a,m}:\mathfrak{R}\longrightarrow\mathfrak{R}$ has a left adjoint.

Proof: We will verify the conditions for Theorem (2.1) and for any ring R we will construct S(R). Let R be any ring, and as the first candidate for S(R) consider $\coprod_{r\in R} C$, with $u_r:C\longrightarrow\coprod_{r\in R} C$ the r^{th} injection into the coproduct. There is a natural mapping from $R\longrightarrow[C,\coprod_{r\in R} C]$ defined by $r\mapsto u_r$ but this will not be a ring homomorphism in general. Therefore, consider the ideal I of $\coprod_{r\in R} C$ generated by all elements of the form:

$$u_{r+s}(c) - \langle u_r,u_s\rangle a(c)$$

$$\text{or} \quad u_{rs}(c) - \langle u_r,u_s\rangle m(c)$$

where $r,s \in R$ and $c \in C$. Let $\rho:\coprod_{r\in R} C\longrightarrow\coprod_{r\in R} C/I$ be the canonical quotient map, and define $S(R) = \coprod_{r\in R} C/I$ and define $v_R:R\longrightarrow[C,S(R)]$ by $v_R(r) = \rho u_r$. To see that v_R is a ring homomorphism, note that if $c \in C$, then

$$(v_R(r) + v_R(s))(c) = \langle v_R(r), v_R(s)\rangle a(c) =$$

$$= \langle \rho u_r, \rho u_s\rangle a(c) = \rho\langle u_r, u_s\rangle a(c) =$$

$$= \rho u_{r+s}(c) = v_R(r + s)(c)$$

so $v_R(r) + v_R(s) = v_R(r + s)$ and similarly for multiplication.

We must now show that if $\Phi: R \longrightarrow [C,B]_{a,m}$ is a ring homomorphism, then there is a unique $\Phi': S(R) \longrightarrow B$ such that

$$\begin{tikzcd} R \arrow[r, "v_R"] \arrow[dr, "\Phi"'] & {[C,S(R)]} \arrow[d, "{[C,\Phi']}"] \\ & {[C,B]} \end{tikzcd}$$

commutes.

Since for each $r \in R$, $\Phi(r): C \longrightarrow B$, there is a mapping $\bar{\Phi}: \coprod_{r\in R} C \longrightarrow B$ such that $\bar{\Phi} u_r = \Phi(r)$ for each r. If $I \subset \ker\bar{\Phi}$, then $\bar{\Phi}$ induces $\Phi': \coprod_{r\in R} C/_{I} \longrightarrow B$. Consider a generator for I, say $u_{r+s}(c) - \langle u_r, u_s\rangle a(c)$.

$$\bar{\Phi}(u_{r+s}(c) - \langle u_r, u_s\rangle a(c)) = \bar{\Phi} u_{r+s}(c) - \bar{\Phi}\langle u_r, u_s\rangle a(c)$$

$$= \Phi(r + s)(c) - \langle \bar{\Phi} u_r, \bar{\Phi} u_s\rangle a(c)$$

$$= \Phi(r + s)(c) - \langle \Phi(r), \Phi(s)\rangle a(c)$$

$$= 0$$

since Φ is a homomorphism. Thus, $I \subset \ker\bar{\Phi}$ and we can

define $\Phi': \coprod_{r\in R} C/_I \longrightarrow B$ by $\Phi'(\rho(x)) = \bar{\Phi}(x)$. To see that Φ' makes the appropriate diagram commute, note that if $r \in R$,

$$[C,\Phi']v_R(r) = \Phi'\rho u_r = \bar{\Phi}u_r = \Phi(r).$$

If $\Phi^*: S(R) \longrightarrow B$ also makes the diagram commute, then $\Phi^*\rho u_r = \Phi'\rho u_r$ for all r, so $\Phi^*\rho = \Phi'\rho$, and since ρ is onto, $\Phi^* = \Phi'$. Thus, Φ' is unique, and, by Theorem (2.1), $[C,-]_{a,m}$ has a left adjoint.

We now know that a representable functor has a left adjoint, but, in general, it is quite difficult to determine precisely what the left adjoint of a particular functor looks like. The following are a few examples where the left adjoint can be rather easily found.

(2.5) Example: Let $\mathcal{C}$ be a category with products and coproducts, and let I be a set. Then the functor T on $\mathcal{C}$ defined by $T(A) = \prod_{i\in I} A$ and $T(f) = \prod_{i\in I} f$ has a left adjoint, namely S, where $S(A) = \coprod_{i\in I} A$ and $S(f) = \coprod_{i\in I} f$.

Proof: For any objects A,B in $\mathcal{C}$, define $\eta_{B,A}: [S(B),A] \longrightarrow [B,T(A)]$ as follows: If $f: \coprod_{i\in I} B \longrightarrow A$, then $fu_i: B \longrightarrow A$ for each i. These morphisms induce a unique morphism $(fu_i): B \longrightarrow \prod_{i\in I} A$. Then define $\eta_{B,A}(f) = (fu_i)$. That η is a natural equivalence of bifunctors is an easy exercise in category theory.

If $\mathcal{C} = \mathfrak{R}$, the category of rings, then the functor T defined above is naturally equivalent to the contracted semigroup functor T_S, where S is a semigroup (with zero) of orthogonal idempotents indexed by the set I.

(2.6) Example: Define the functor T on $\mathfrak{R}$ by $T(R) = R_0$, where R_0 has the same additive group as R, but has trivial multiplication. Then T has a left adjoint S where $S(A) = F(A/A^2)$ and $F(G)$ denotes the tensor ring on the abelian group G.

Proof: The category of abelian groups, Ab, can be considered as a subcategory of $\mathfrak{R}$, namely the full subcategory consisting of all rings with trivial multiplication. The functor T can then be factored as

$$\mathfrak{R} \xrightarrow{U} Ab \xrightarrow{I} \mathfrak{R}$$

where U is the forgetful functor and I is the inclusion functor. F, the functor which sends an abelian group to its tensor ring, is the left adjoint of U (See [3]). We will show that I has a left adjoint J where $J(R) = R/R^2$ and if $f:R \longrightarrow R'$ then $J(f):R/R^2 \longrightarrow R'/R'^2$ is defined by $J(f)(r + R^2) = f(r) + R'^2$. Let R be a ring and G an abelian group. Define $\eta_{G,R}:[J(R),G] \longrightarrow [R,I(G)]$ by $\eta_{G,R}(\alpha)(r) = \alpha(r + R^2)$ for any $\alpha \in [J(R),G]$. $\eta_{G,R}$ is clearly one-to-one. It is also onto, for suppose $\beta:R \longrightarrow G$. Since G has trivial multiplication, $R^2 \subset \ker\beta$, so $\bar{\beta}:R/R^2 \longrightarrow G$ defined by $\bar{\beta}(r + R^2) = \beta(r)$ is well defined and $\eta_{G,R}(\bar{\beta}) = \beta$. It is easily verified that η is

natural in both variables so that J is the left adjoint of I. The left adjoint of $T = IU$ is FJ, which is the functor S defined above.

If we note that the functor T is naturally equivalent to the contracted semigroup functor T_S where S is the two-element semigroup with zero multiplication, then we see that the coring representing T is $(W(x),a,m)$ where $W(x)$ is the free ring on one generator x with $a(x) = u_1(x) + u_2(x)$ and $m(x) = 0$.

(2.7) Example: For each fixed natural number n, define a functor T_n on $\mathcal{R}$ by letting $T_n(R)$ denote the subring of R of all elements of additive order n. If $f:R \longrightarrow R'$, then $T_n(f) = f|T_n(R)$. Then T_n has a left adjoint S_n where $S_n(R) = R/nR$ and if $f:R \longrightarrow R'$, then $S_n(f)(r + nR) = f(r)+nR'$.

Proof: If R and R' are rings, define $\eta_{R',R}:[S_n(R),R'] \longrightarrow [R,T_n(R')]$ by $\eta_{R',R}(f)(r) = f(r + nR)$. It is clear that $\eta_{R',R}$ is one-to-one. It is also onto, for suppose $h:R \longrightarrow T_n(R')$. $h(nr) = nh(r) = 0$, so $nR \subset \ker h$, so define $f:R/nR \dashrightarrow R'$ by $f(r + nR) = h(r)$. Then $\eta_{R',R}(f) = h$. That η is natural in both variables is easily verified.

The coring which represents T_n will be discussed later (3.12).

(2.8) Example: Let S be an infinite semigroup with 0

and let F_S be the functor such that $F_S(R) = R_0[S]$ and $F_S(f)(\Sigma r_s s) = \Sigma f(r_s)s$. Then F_S does not have a left adjoint.

<u>Proof</u>: We will show that F_S does not preserve products. Suppose $p_i : \prod_{i=1}^{\infty} Z \longrightarrow Z$ is the product of a countable collection of copies of the integers, and $q_i : \prod_{i=1}^{\infty} Z_0[S] \longrightarrow Z_0[S]$ is the product of copies of $Z_0[S]$. If F_S preserves products, then $F_S(p_i) : (\prod_{i=1}^{\infty} Z)_0[S] \longrightarrow Z_0[S]$ is also a product of copies of $Z_0[S]$ and there exists an isomorphism $\xi : \prod_{i=1}^{\infty} Z_0[S] \longrightarrow (\prod_{i=1}^{\infty} Z)_0[S]$ such that $F_S(p_i)\xi = q_i$ for all i. Let $s_1, s_2, \ldots$ be a countable subset of S. Then $x = (s_1, s_2, s_3, \ldots) \in \prod_{i=1}^{\infty} Z_0[S]$, and $F_S(p_i)\xi(x) = q_i(x) = s_i$ for each i. If $\xi(x) = \sum_{i=1}^{n} (n_{j1}, n_{j2}, \ldots) t_j$, then $F_S(p_i)\xi(x) = \sum_{i=1}^{n} n_{ji} t_j = s_i$ for each i. So $t_i = s_i$ and $n_{ji} = \delta_{ji}$ for each i. But this is impossible since there is only a finite number of the t_i and an infinite number of the s_i. Thus, F_S cannot preserve products.

CHAPTER III

STRUCTURE OF CORINGS

Kan [4] showed that the only comonoids in the category of groups are free groups with a more or less trivial comultiplication defined as follows: There exists a free basis X for the comonoid C such that $m(x) = u_1(x)u_2(x)$ for each $x \in X$.

Things are not so simple for corings in $\mathfrak{R}$, however. In the last chapter we saw examples of corings which are free rings with nontrivial comultiplications. There are also examples of corings in $\mathfrak{R}$ which are not free. For example, if (C,a,m) is a coring where C is free, and if S_n is the functor defined in Example (2.8), then by Theorem (2.4), $(S_n(C),S_n(a),S_n(m))$ is also a coring. But $S_n(C) = C/nC$ which cannot be a free ring since its additive group is torsion.

We now have a large number of examples of corings which are not free rings. However, if we let $\mathfrak{A}_K$ denote the category of K-algebras where K is a field, then it is not known if there exist examples of non-free corings in $\mathfrak{A}_K$. In fact, even though we do not know if a co-abelian group in $\mathfrak{A}_K$

must be free, we will show that a co-abelian group in $\mathcal{C}_K$ must satisfy many of the properties of a free algebra. First we need to know something about coproducts in $\mathcal{C}_K$.

(3.1) <u>Construction of coproducts in $\mathcal{C}_K$</u>: This construction is alluded to by Cohn [3] and outlined in more detail by Bergman [5]. This construction can be used to construct coproducts in $\mathcal{R}$ if the rings involved have free additive groups.

Let A_1 and A_2 be K-algebras, and let X_i be a basis for A_i as a K-vector space. Assuming A_1 and A_2 are disjoint, let $X_1 \cup X_2 = \{x_\nu, \nu \in N\}$. For each finite sequence $J = (\nu_1, \ldots, \nu_n)$ of elements of N, define the <u>monomial</u> $x_J = x_{\nu_1} x_{\nu_2} \ldots x_{\nu_n}$. Two monomials x_I, x_J will be equal if and only if the sequences I and J are equal. We will say that a monomial $x_J = x_{\nu_1} x_{\nu_2} \ldots x_{\nu_n}$ is <u>proper</u> if no two successive x_{ν_i} belong to the same X_k, k=1,2. Let A be the K-vector space whose basis consists of all proper monomials.

We will define a multiplication on basis elements of A as follows: Let $x_I = x_{\mu_1} x_{\mu_2} \ldots x_{\mu_m}$, $x_J = x_{\nu_1} x_{\nu_2} \ldots x_{\nu_n}$ be proper monomials.

Case 1: If x_{μ_m} and x_{ν_1} belong to different X_i, then $x_I x_J = x_{\mu_1} x_{\mu_2} \ldots x_{\mu_m} x_{\nu_1} x_{\nu_2} \ldots x_{\nu_n}$.

Case 2: If x_{μ_m} and x_{ν_1} belong to the same X_i, then $x_{\mu_m}, x_{\nu_1} \in R_i$, so x_{μ_m} and x_{ν_1} can be multiplied to give $x_{\mu_m} x_{\nu_1} = \sum_{j=1}^{s} k_j x_j$. Then define

$$x_I x_J = \sum_{j=1}^{s} k_j x_{\mu_1} x_{\mu_2} \cdots x_{\mu_{m-1}} x_j x_{\nu_2} x_{\nu_3} \cdots x_{\nu_n}.$$

With this multiplication A will be a K-algebra and will be the coproduct of A_1 and A_2 in $\mathcal{C}_K$.

(3.2) <u>Lemma</u>: Let $\mathcal{C}$ be a category with zero maps, and let (C,m) be a cogroup in $\mathcal{C}$. Then if A is any object in $\mathcal{C}$, the zero map from C to A is the identity element of $[C,A]_m$.

<u>Proof</u>: Let $0:C \longrightarrow A$ be the zero map, and let $e:C \longrightarrow A$ be the identity element of $[C,A]_m$. Since e is the identity of $[C,A]_m$, we know $0 \cdot e = 0$. But $0 \cdot 0 = \langle 0,0 \rangle m = 0$ (since $\langle 0,0 \rangle$ is the zero map from $C*C$ to A). Thus, $0 \cdot e = 0 \cdot 0$, so $e = 0$.

(3.3) Lemma: If (C,m) is a comonoid in a category $\mathcal{C}$, then m must be a monomorphism.

<u>Proof:</u> Since (C,m) is a comonoid in $\mathcal{C}$, $[C,C]_m$ is a monoid and must have an identity e such that $\langle f,e \rangle m = f$ for any $f \in [C,C]$. In particular, if $1:C \longrightarrow C$ is the identity map then $\langle 1,e \rangle m = 1$. Since 1 is a monomorphism, m must also be a monomorphism.

(3.4) <u>Lemma</u>: If $m:C \longrightarrow C*C$ is a co-operation in a category $\mathcal{C}$, then m is commutative if and only if $\tau m = m$ where $\tau = \langle u_2, u_1 \rangle : C*C \longrightarrow C*C$.

Proof: If m is commutative, then $[C,C*C]_m$ is commutative, so $u_1 \cdot u_2 = u_2 \cdot u_1$ or $<u_1,u_2>m = <u_2,u_1>m$. But $<u_1,u_2>$ is the identity map on $C*C$, so $m = \tau m$.

Conversely, suppose $m = \tau m$. Let X be any object in $\mathcal{C}$ and let f,g be elements of $[C,X]$. Then $f \cdot g = <f,g>m =$

$$= <f,g>\tau m = <f,g><u_2,u_1>m = <<f,g>u_2,<f,g>u_1>m =$$

$$= <g,f>m = g \cdot f.$$

Thus m is commutative.

A free ring has a degree function deg defined on it satisfying the following properties.

1. $\deg(x)$ is a positive integer if $x \neq 0$; $\deg(0) = -\infty$.
2. $\deg(x-y) \leq \max\{\deg(x),\deg(y)\}$.
3. $\deg(xy) = \deg(x) + \deg(y)$.

A ring with such a degree function is called a <u>valuation ring</u>.

(3.5) <u>Proposition</u>: Let (C,a) be a co-abelian group in $\mathcal{C}_K$ where K is a field. Then C is a valuation ring with no elements of degree zero.

<u>Proof</u>: Let $\{x_\nu, \nu \in N\}$ be a K-vector space basis for C. $C*C$ contains two disjoint copies of C, so let $\{x_{1,\nu}, \nu \in N\}$ be a basis for the first copy, and let $\{x_{2,\nu}, \nu \in N\}$ be a basis for the second. Then $u_i(x_\nu) = x_{i,\nu}$, $i = 1,2$. Then $C*C$ has as a basis the set of all proper monomials in $Y = \{x_{1,\nu} | \nu \in N\} \cup \{x_{2,\nu} | \nu \in N\}$.

A degree function deg can be defined on C as follows: If c is a nonzero element of C, since a is a monomorphism (Lemma (3.6)), $a(c) \neq 0$. $a(c) = \sum_{i=1}^{n} k_i x_{I_i}$, where each x_{I_i} is is a proper monomial from Y. Define deg(c) to be the maximal length of the monomials x_{I_i}, where the length of a monomial $x_{\nu_1} x_{\nu_2} \ldots x_{\nu_p}$ is just p. It is clear from the vector space addition in C*C that $\deg(c-d) \leq \max\{\deg(c), \deg(d)\}$.

It remains to be shown that $\deg(cd) = \deg(c) + \deg(d)$. Let c,d be nonzero elements of C, and suppose $\deg(c) = p$, $\deg(d) = q$. $a(c) = \sum_{i=1}^{n} k_i x_{I_i}$, $a(d) = \sum_{j=1}^{m} k_j' x_{J_j}$. Then one of the x_{I_i}, say $x_I = x_{i_1\nu_1} x_{i_2\nu_2} \ldots x_{i_p\nu_p}$ must have length p, and one of the x_{J_j}, say $x_J = x_{j_1\mu_1} x_{j_2\mu_2} \ldots x_{j_q\mu_q}$ must have length q. If $i_p \neq j_1$, then $x_{i_p\nu_p}$ and $x_{j_1\mu_1}$ come from different copies of C in C*C, so $x_I x_J = x_{i_1\nu_1} \ldots x_{i_p\nu_p} x_{j_1\mu_1} \ldots x_{j_q\mu_q}$ and has length p+q. On the other hand, suppose $i_p = j_1$. Then $x_{i_p\nu_p}$ and $x_{j_1\mu_1}$ are from the same copy of C and, when multiplied, $x_I x_J$ would not give a monomial of length p+q. However, a is commutative and by Lemma (3.4), $\tau a = a$. $\tau(x_J) = x_{J'} = x_{j_1'\mu_1} x_{j_2'\mu_2} \ldots x_{j_q'\mu_q}$ where $j_i' \neq j_i$ for $i=1,\ldots q$. (In other words, to get $x_{J'}$, replace a factor of x_J of the

form $x_{1\mu}$ by $x_{2\mu}$ and vice versa.) Since $\tau a = a$, $a(c)$ must contain $x_{J'}$ as one of its summands. $x_I x_{J'} =$ $x_{i_1\nu_1} x_{i_2\nu_2} \cdots x_{i_p\nu_p} x_{j_1'\mu_1} x_{j_2'\mu_2} x_{j_3'\mu_3} \cdots x_{j_q'\mu_q}$ which has length $p + q$. In either case ($i_p \neq j_1$, $i_p = j_1$) $a(c)a(d)$ yields a monomial of length $p + q$. It is easy to see that $x_I x_J$ could not be obtained by multiplying two monomials $x_K x_L$ of lengths $\leq p,q$ respectively unless $x_K = x_I$ and $x_J = x_L$. Thus, $x_I x_J$ (or $x_I x_{J'}$, in case $i_p = j_1$) will not be cancelled out by another monomial of the same length. Thus, $\deg(cd) = p + q$, and deg is a valuation. By the way deg is defined, there can be no elements of C of degree 0.

Corollary: A co-abelian group in $\mathcal{G}_K$, K a field, can have no zero divisors or idempotents.

Note: The above result holds for corings C in the category of rings as long as the additive group of C is a free abelian group.

Let (C,a,m) be a coring in $\mathcal{R}$ where C is the free ring on some set X. We say that the coaddition is standard if $a(x) = u_1(x) + u_2(x)$ for each x in X. Note that all the examples previously mentioned of corings on free rings (corings representing semigroup functors, etc.) have had standard coaddition. In fact, it is conjectured that any

coaddition on a free ring must be standard (for some suitable free basis of the ring).

If C is the free ring on the set $X = \{x_\nu, \nu \in N\}$, then $C*C$ can be thought of as the free ring on $X \cup Y$ where $Y = \{y_\nu, \nu \in N\}$ and $x \cap Y = \emptyset$. The coproduct injections are defined by $u_1(x_\nu) = x_\nu$, $u_2(x_\nu) = y_\nu$ for each $\nu \in N$.

The following proposition shows the structure of comultiplication in standard corings.

(3.6) Proposition: If (C,a,m) is a coring in $\mathcal{R}$ where C is free on $X = \{x_\nu, \nu \in N\}$ and a is standard, then m must have the following form for each $\nu \in N$:

$$m(x_\nu) = \sum_{i=1}^{k} n_i p_i$$

where either $p_i = x_{\mu_i} y_{\nu_i}$ or $p_i = y_{\nu_i} x_{\mu_i}$ and $n_i \in Z$.

Proof: $m(x_\mu)$ is in the free algebra on $X \cup Y$, so $m(x_\mu)$ is a "word" in the x_ν's and y_ν's. We will denote $m(x_\mu)$ by $w_\mu(x_\nu, y_\nu)$. If A is any ring and $f,g: C \longrightarrow A$, then

$$(f + g)(x_\mu) = \langle f,g \rangle a(x_\mu) = f(x_\mu) + g(x_\mu)$$

$$(f \cdot g)(x_\mu) = \langle f,g \rangle m(x_\mu) = w_\mu(f(x_\nu), g(x_\nu)).$$

If we consider the ring $[C,C]_{a,m}$ and the fact that $1_C \cdot 0 = 0$, then we see that $w_\mu(1_C(x_\nu), 0(x_\nu)) = w_\mu(x_\nu, 0) = 0$. But $w_\mu(x_\nu, 0)$ consists of precisely those terms of $w_\mu(x_\nu, y_\nu)$ which do not contain any elements from Y as factors. Since

$w_\mu(x_\nu,0) = 0$, we must conclude that each term in the "word" $w_\mu(x_\nu,y_\nu)$ must contain at least one of the y_ν. (Or else $w_\mu = 0$ which satisfies the conclusion of the proposition.) A similar argument shows that each term must contain at least one of the x_ν.

We have now shown that each term of $w_\mu(x_\nu,y_\nu)$ contains at least one of the x_ν and one of the y_ν. All that remains is to show that each term does not contain more than one. Suppose that a term of $w_\mu(x_\nu,y_\nu)$ contains more than one of the x_ν, for example. Let $V = \{v_\nu, \nu \in N\}$, and let R be the free ring on $X \cup Y \cup V$. Define $f_1,f_2,f_3 : C \longrightarrow R$ by $f_1(x_\nu) = x_\nu$, $f_2(x_\nu) = y_\nu$, $f_3(x_\nu) = v_\nu$ for each ν. Since $[C,R]_{a,m}$ is a ring, the distributive laws hold, and for each $\mu \in N$,

$$((f_1 + f_2)\cdot f_3)(x_\mu) = (f_1f_3 + f_2f_3)(x_\mu)$$

$$w_\mu((f_1 + f_2)(x_\nu), f_3(x_\nu)) = w_\mu(f_1(x_\nu), f_3(x_\nu)) + w_\mu(f_2(x_\nu), f_3(x_\nu))$$

$$w_\mu(x_\nu + y_\nu, v_\nu) = w_\mu(x_\nu, v_\nu) + w_\mu(y_\nu, v_\nu). \qquad *$$

To get the left hand side of *, replace each x_ν in $w_\mu(x_\nu,y_\nu)$ by $x_\nu + y_\nu$, and replace each y_ν by v_ν. But if a term in $w_\mu(x_\nu,y_\nu)$ contains more than one of the x_ν, when they are replaced by the appropriate $x_\nu + y_\nu$ and multiplied, some terms would result which would contain at least one x_ν, one y_ν, and one v_ν. (For example, suppose $w_\mu(x_\nu,y_\nu)$ contains

the term $x_{\nu_1} y_{\nu_2} x_{\nu_3}$. Substituting, we get $(x_{\nu_1} + y_{\nu_1}) v_{\nu_2} (x_{\nu_3} + y_{\nu_3})$, and expanding results in the term $x_{\nu_1} v_{\nu_2} y_{\nu_3}$, among others.) It is easily seen that two terms containing an x_ν, a y_ν, and a v_ν cannot cancel each other out. The right hand side cannot contain any terms with an x_ν, y_ν, and v_ν. Thus * is impossible and we must conclude that no term in $w_\mu(x_\nu, y_\nu)$ can contain more than one x_ν, and, by a similar argument, could contain no more than one of the y_ν. Thus, either $w_\mu(x_\nu, y_\nu) = 0$ or each term contains exactly one of the x_ν and one of the y_ν, and $w_\mu(x_\nu, y_\nu)$ must be of the required form.

The functor $A \otimes -$ plays an important role in the theory of adjoint functors in the category of abelian groups. The tensor product of rings does not have the same role in the category of rings, but it does have some relation to adjoint functors. In the following we will investigate the relation of tensor product of rings to adjoint functors on the category of rings.

(3.7) <u>Lemma</u>: Let (C,a,m) be a coring where C is free on $\{x_1, \dots x_n\}$ and a is standard. Let Z be the ring of integers, and for $i = 1, \dots, n$, define $f_i : C \longrightarrow Z$ by

$f_i(x_j) = \delta_{ij}$. Then, as an abelian group, $[C,Z]_{a,m}$ is free on $\{f_1, f_2, \ldots, f_n\}$.

Proof: Let G be the free abelian group on the generators $v_1, v_2, \ldots, v_n$. Define $\Phi : [C,Z]_{a,m} \longrightarrow G$ by $\Phi(h) = \sum_{i=1}^{n} h(x_i)v_i$. That Φ is one-to-one and onto is clear from the fact that C is free. We claim that Φ is an additive isomorphism:

$$\Phi(h + g) = \Phi(\langle h,g\rangle a) = \sum_{i=1}^{n} \langle h,g\rangle a(x_i)v_i =$$
$$= \sum_{i=1}^{n} \langle h,g\rangle (u_1(x_i) + u_2(x_i))v_i = \sum_{i=1}^{n} (h(x_i) + g(x_i))v_i$$
$$= \sum_{i=1}^{n} h(x_i)v_i + \sum_{i=1}^{n} g(x_i)v_i = \Phi(h) + \Phi(g).$$

We now know that $[C,Z]_{a,m}$ is a free abelian group, and a set of generators is $\{\Phi^{-1}(v_i)\}$, $i=1,2,\ldots,n$. But

$$\Phi(f_i) = \sum_{j=1}^{n} f_i(x_j)v_j = \sum_{j=1}^{n} \delta_{ij}v_j = v_i.$$

So $f_i = \Phi^{-1}(v_i)$, and $[C,Z]_{a,m}$ is free on $f_1, f_2, \ldots, f_n$.

Note: In all that follows, all tensor products are over Z, the ring of integers.

(3.8) Proposition: Let (C,a,m) be a coring where C is free on $x_1, x_2, \ldots, x_n$, a is standard, and $m(x_i) = \sum_{j=1}^{n} \sum_{k=1}^{n} z_{ijk} u_1(x_j) u_2(x_k)$ for each i, where $z_{ijk} \in Z$. Then the functor $[C,-]_{a,m}$ is naturally equivalent to $[C,Z]_{a,m} \otimes -$.

Proof: From Lemma (3.7) we know that any element of

$[C,Z]_{a,m}$ can be written as:

$$\sum_{j=1}^{m}[\sum_{i=1}^{n} z_{ji} f_i) \otimes r_j] = \sum_{j=1}^{m}[\sum_{i=1}^{n}(z_{ji} f_i \otimes r_j)]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (z_{ji} f_i \otimes r_j) = \sum_{i=1}^{n}[\sum_{j=1}^{m} f_i \otimes z_{ji} r_j]$$

$$= \sum_{i=1}^{n} f_i \otimes \sum_{j=1}^{m} z_{ji} r_j = \sum_{i=1}^{n} f_i \otimes r_i'.$$

We need to show that for any ring R there is a natural isomorphism $\Phi_R:[C,Z]_{a,m} \otimes R \longrightarrow [C,R]_{a,m}$. If $x \in [C,Z]_{a,m} \otimes R$, then $\Phi_R(x)$ is a function in $[C,R]$ and it is sufficient to define it for the basis elements of C. Define Φ_R by $\Phi_R(\sum_{i=1}^{n} z_i f_i \otimes r)(x_j) = z_j r$. Is is routine to verify that Φ_R is well defined and is an additive homomorphism.

Note that $f_j \cdot f_k = \sum_{i=1}^{n} z_{ijk} f_i$ since

$$(f_j \cdot f_k)(x_i) = \langle f_j, f_k \rangle m(x_i) =$$

$$= \langle f_j, f_k \rangle (\sum_{\nu=1}^{n} \sum_{\mu=1}^{n} z_{i\nu\mu} u_1(x_\nu) u_2(x_\mu))$$

$$= \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} z_{i\nu\mu} f_j(x_\nu) f_k(x_\mu) = z_{ijk}$$

$$= (\sum_{i=1}^{n} z_{ijk} f_i)(x_i).$$

Also note that if y is an arbitrary element of $[C,Z]_{a,m} \otimes R$, then y can be written as $\sum_{i=1}^{n} f_i \otimes r_i$, and Φ_R is the function from C to R defined by $\Phi_R(y)(x_i) = r_i$.

We now show that Φ_R is a multiplicative homomorphism. If x_p is one of the basis elements of C, then

$$\Phi_R[(\sum_{i=1}^{n} f_i \otimes r_i)(\sum_{i=1}^{n} f_i \otimes r_i')]\ (x_p) =$$

$$= \Phi_R[\sum_{j=1}^{n}\sum_{k=1}^{n} (f_j \otimes r_j)(f_k \otimes r_k')]\ (x_p)$$

$$= \Phi_R[\sum_{j=1}^{n}\sum_{k=1}^{n} (f_j f_k \otimes r_j r_k')]\ (x_p)$$

$$= \Phi_R[\sum_{j=1}^{n}\sum_{k=1}^{n}(\sum_{i=1}^{n} z_{ijk} f_i \otimes r_j r_k')]\ (x_p)$$

$$= \Phi_R[\sum_{i=1}^{n}(f_i \otimes \sum_{j=1}^{n}\sum_{k=1}^{n} z_{ijk} r_j r_k')]\ (x_p)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} z_{pjk} r_j r_k'.$$

$$\Phi_R(\sum_{i=1}^{n} f_i \otimes r_i)\ \Phi_R(\sum_{i=1}^{n} f_i \otimes r_i')\ (x_p) =$$

$$= <\Phi_R(\sum_{i=1}^{n} f_i \otimes r_i), \Phi_R(\sum_{i=1}^{n} f_i \otimes r_i')>m(x_p)$$

$$= <\Phi_R(\sum_{i=1}^{n} f_i \otimes r_i), \Phi_R(\sum_{i=1}^{n} f_i \otimes r_i')>(\sum_{j=1}^{n}\sum_{k=1}^{n} z_{pjk} u_1(x_j) u_2(x_k))$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} z_{pjk} \Phi_R(\sum_{i=1}^{n} f_i \otimes r_i)(x_j)\Phi_R(\sum_{i=1}^{n} f_i \otimes r_i')(x_k)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} z_{pjk} r_j r_k'.$$

Thus, Φ_R is a ring homomorphism.

Φ_R is onto, for if $h:C \longrightarrow R$, then $\Phi_R(\sum_{i=1}^{n} f_i \otimes h(x_i)) = h$.

Φ_R is also one-to-one, for if $\Phi_R(\sum_{i=1}^{n} f_i \otimes r_i) = 0$, then $\Phi_R(\sum_{i=1}^{n} f_i \otimes r_i)(x_j) = 0$ for all j, so $r_j = 0$ for all j, and $\sum_{i=1}^{n} f_i \otimes r_i = 0$. Thus Φ_R is an isomorphism.

All that remains is to show that Φ is natural. Suppose $h:R \longrightarrow R'$. We must show that the following diagram is commutative:

$$\begin{array}{ccc} [C,Z]_{a,m} \otimes R & \xrightarrow{\Phi_R} & [C,R]_{a,m} \\ \Big\downarrow 1 \otimes h & & \Big\downarrow [C,h] \\ [C,Z]_{a,m} \otimes R' & \xrightarrow[\Phi_{R'}]{} & [C,R']_{a,m} \end{array}$$

$$[C,h]\ \Phi_R\ (\sum_{i=1}^{n} f_i \otimes r_i)\ (x_j) = h\ \Phi_R\ (\sum_{i=1}^{n} f_i \otimes r_i)\ (x_j)$$
$$= h(r_j).$$

$$\Phi_{R'}\ (1 \otimes h)\ (\sum_{i=1}^{n} f_i \otimes r_i)\ (x_j) = h\Phi_R(\sum_{i=1}^{n} f_i \otimes h(r_i))(x_j).$$
$$= h(r_j).$$

Thus, Φ is a natural equivalence.

The following lemma uses some well-known results about Ab, the category of abelian groups. For complete statements and proofs of these results, the reader is referred to [6].

(3.9) Lemma: Let G be an abelian group such that the functor $G \otimes -: Ab \longrightarrow Ab$ has a left adjoint. Then G is free and of finite rank.

Proof: $G \otimes -$ has a right adjoint for any group G. Suppose it also has a left adjoint. Any functor on Ab with a left adjoint must be naturally equivalent to $[H,-]$ for some abelian group H. Thus, $G \otimes - \cong [H,-]$. $G \otimes -$ is an

exact functor, but $[H,-]$ is exact if and only if H is projective, so H must be a free abelian group. Suppose H is free on an infinite set X. $G \otimes -$ has a right adjoint, so $[H,-]$ has a right adjoint and must preserve colimits, direct sums in particular. If we let G_i be the cyclic group of order i, i=1,2,..., then, since $[H,-]$ preserves direct sums, $[H,\sum_{i=1}^{\infty} G_i] \cong \sum_{i=1}^{\infty}[H,G_i]$. Since H is free on X, $\sum_{i=1}^{\infty}[H,G_i] \cong \sum_{i=1}^{\infty}(\prod_{x\in X} G_i)$, a torsion group. On the other hand, $[H,\sum_{i=1}^{\infty} G_i] \cong \prod_{x\in X}(\sum_{i=1}^{\infty} G_i)$ which contains some torsion free elements if X is infinite, a contradiction. Thus, H is free on a finite set. Since $G \cong G \otimes Z \cong [H,Z] \cong H$, G must also be free and of finite rank.

(3.10) Theorem: If R is a ring, then the functor $R \otimes -:\mathcal{R} \longrightarrow \mathcal{R}$ has a left adjoint if and only if the additive group of R is free and of finite rank.

Proof: Suppose $R \otimes -$ has a left adjoint. The category Ab can be considered as a subcategory of $\mathcal{R}$, namely the full subcategory of all rings with trivial multiplication. If G is a ring with trivial multiplication, then $R \otimes G$ will also have trivial multiplication, so $R \otimes -$ can be considered as a functor from Ab to Ab. Also, since limits in Ab are the same as the corresponding limits in $\mathcal{R}$, and since $R \otimes -$ is limit preserving on $\mathcal{R}$, it follows that $R \otimes -:Ab \longrightarrow Ab$ is a limit-preserving functor. Since Ab is complete, locally

small, and has a cogenerator, by the Special Adjoint Functor Theorem, $R \otimes -:Ab \longrightarrow Ab$ has a left adjoint. From Lemma (3.10), R, as an abelian group, must be free and of finite rank.

Conversely, suppose R is free and of finite rank as an abelian group, and let $b_1, b_2, \ldots, b_n$ be a basis for the additive group of R. Then there exist integers n_{ijk} such that $b_j b_k = \sum_{i=1}^{n} n_{ijk} b_i$.

Let C be the free ring on $x_1, x_2, \ldots, x_n$, and define co-operations $a, m: C \longrightarrow C*C$ by

$$a(x_i) = u_1(x_i) + u_2(x_i)$$

$$m(x_i) = \sum_{j=1}^{n} \sum_{k=1}^{n} n_{ijk} u_1(x_j) u_2(x_k).$$

We claim (C,a,m) is a coring and that $[C,-]_{a,m} \cong R \otimes -$. By Proposition (3.8) it is sufficient to show $[C,Z]_{a,m} \cong R$.

Define $\xi:[C,Z]_{a,m} \longrightarrow R$ by $\xi(f) = \sum_{i=1}^{n} f(x_i) b_i$.

$$\xi(f+g) = \sum_{i=1}^{n} (f+g)(x_i) b_i = \sum_{i=1}^{n} \langle f,g \rangle a(x_i) b_i$$

$$= \sum_{i=1}^{n} \langle f,g \rangle (u_1(x_i) + u_2(x_i)) b_i$$

$$= \sum_{i=1}^{n} [f(x_i) + g(x_i)] b_i = \sum_{i=1}^{n} f(x_i) b_i + \sum_{i=1}^{n} g(x_i) b_i$$

$$= \xi(f) + \xi(g).$$

$$\xi(fg) = \sum_{i=1}^{n} (fg)(x_i) b_i = \sum_{i=1}^{n} \langle f,g \rangle m(x_i) b_i$$

$$= \sum_{i=1}^{n} \langle f,g \rangle \left(\sum_{j=1}^{n} \sum_{k=1}^{n} n_{ijk} u_1(x_j) u_2(x_k) \right) b_i$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} n_{ijk} f(x_j) g(x_k) b_i.$$

$$\Phi(f)\Phi(g) = (\sum_{i=1}^{n} f(x_i)b_i)\ (\sum_{i=1}^{n} g(x_i)b_i)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} f(x_j)g(x_k)b_jb_k$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} [f(x_j)g(x_k)(\sum_{i=1}^{n} n_{ijk}b_i)]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} n_{ijk}f(x_j)g(x_k)b_i.$$

Since Φ is clearly one-to-one and onto, Φ is an isomorphism. Thus, $[C,Z]_{a,m}$ is a ring and $[C,-]_{a,m} \cong [C,Z]_{a,m} \otimes - \cong R \otimes -$. Since $R \otimes -$ is representable, it has a left adjoint.

Corollary: Let $T:\mathcal{R} \longrightarrow \mathcal{R}$ have a left adjoint. Then $T \cong R \otimes -$ if and only if T is representable by a coring (C,a,m) where C is free on $x_1,x_2,\ldots,x_n$, a is standard, and

$$m(x_i) = \sum_{j=1}^{n} \sum_{k=1}^{n} n_{ijk}u_1(x_j)u_2(x_k).$$

Proof: This follows from Proposition (3.8) and the proof of the above theorem where it was found that the coring representing $R \otimes -$ is as stated in the corallary.

In light of the construction mentioned in (1.2), it is seen that if S is a finite semigroup with zero, then the semigroup ring functor T_S is naturally equivalent to $Z_0[S] \otimes -$. Not every functor representable by a finitely generated free coring is naturally equivalent to a tensor product functor, however. An example of such a functor is the opposite functor T_{op} which sends a ring into its opposite ring. $T_{op} \cong [C,-]_{a,m}$ where (C,a,m) is a standard coring

on one generator x, and $m(x) = u_2(x)u_1(x)$. If $T_{op} \cong R \otimes -$, then $T_{op}(Z) \cong R \otimes Z \cong R$. But $T_{op}(Z) = Z$, and $Z \otimes -$ is not naturally equivalent to T_{op}.

In the category of abelian groups, the functor $A \otimes -$ always has a right adjoint. There are rings such that the functor $R \otimes -$ on the category of rings has a right adjoint, but this is not true for every ring R.

(3.11) Example: Denote by Z_n the ring of integers modulo n. Then $Z_n \otimes -:\mathfrak{R} \longrightarrow \mathfrak{R}$ has a right adjoint.

Proof: We will show that $Z_n \otimes -$ is naturally equivalent to the functor S_n described in Example (2.8), where $S_n(R) = R/nR$. If R is a ring, define $\Phi_R:R/nR \longrightarrow Z_n \otimes R$ by $\Phi_R(r + nR) = 1 \otimes r$. It is easily verified that Φ_R is a well-defined ring homomorphism. To show that Φ_R is one-to-one and onto, we can define $\Psi_R:Z_n \otimes R \longrightarrow R/nR$ by $\Psi_R(z + (n) \otimes r) = zr + nR$. It can be easily shown that $\Psi_R\Phi_R = 1_{R/nR}$ and that $\Phi_R\Psi_R = 1_{Z_n \otimes R}$. It is also a routine calculation to show that Φ is a natural transformation and hence that $S_n \cong Z_n \otimes -$.

From Theorem (2.4) and the above, we know that if (C,a,m) is a coring in $\mathfrak{R}$, then $(Z_n \otimes C, 1 \otimes a, 1 \otimes m)$ will

also be a coring. In particular, if (G,a,m) is the coring representing the identity functor, then $(Z_n \otimes G, 1 \otimes a, 1 \otimes m)$ is the coring representing the functor T_n described in Example (2.8)

(3.12) Example: $Z \times Z \otimes - : \mathfrak{R} \longrightarrow \mathfrak{R}$ does not have a right adjoint.

Proof: The functor T on $\mathfrak{R}$ defined by $T(R) = R \times R$ and $T(f) = f \times f$, discussed in Example (2.7), has a coring representation of the form needed for Proposition (3.8). It follows that T is naturally equivalent to $Z \times Z \otimes -$, and we will show that T cannot have a right adjoint. In particular, we will show that it is not coproduct preserving.

If T is coproduct preserving, then, since $u_i : Z \longrightarrow Z * Z$, $i = 1,2$, defines a coproduct, then $u_i \times u_i : Z \times Z \longrightarrow (Z * Z) \times (Z * Z)$ will also define a coproduct.

$$\begin{tikzcd}
 & (Z * Z) \times (Z * Z) \arrow[dl, "u_1 \times u_1"'] \arrow[dr, "u_2 \times u_2"] \arrow[dd, dashed, "h"] & \\
Z \times Z \arrow[dr, "p_1"'] & & Z \times Z \arrow[dl, "p_2"] \\
 & Z &
\end{tikzcd}$$

If p_1, p_2 represent projectons from the product, there must exist a ring homomorphism, h, making the above diagram commute. Consider the element $(u_1(1), u_2(1)) \in (Z * Z) \times (Z * Z)$.

$(u_1(1), u_2(1)) = (u_1 \times u_1)(1,0) + (u_2 \times u_2)(0,1)$. So

$$h(u_1(1), u_2(1)) = h(u_1 \times u_1)(1,0) + h(u_2 \times u_2)(0,1)$$

$$= p_1(1,0) + p_2(0,1) = 1 + 1 = 2.$$

But $(u_1(1), u_2(1))^2 = (u_1(1), u_2(1))$; but

$h(u_1(1), u_2(1))^2 \neq [h(u_1(1), u_2(1))]^2$, so h cannot be a ring homomorphism. Thus, T cannot be coproduct preserving and cannot have a right adjoint.

CHAPTER IV

THE CATEGORY OF CORINGS

Let $\mathcal{C}$ be a category with coproducts. Let A_1, A_2, B_1, B_2 be objects in $\mathcal{C}$ and let $f_i: A_i \longrightarrow B_i$, $i=1,2$, be morphisms in $\mathcal{C}$. Define $f_1 * f_1: A_1 * A_2 \longrightarrow B_1 * B_2$ as the unique morphism such that the diagram

$$\begin{tikzcd}
 & A_1 * A_2 \arrow[dl, "u_1"'] \arrow[dr, "u_2"] \arrow[ddd, "f_1 * f_2"] & \\
A_1 \arrow[d, "f_1"] & & A_2 \arrow[d, "f_2"'] \\
B_1 \arrow[dr, "u_1'"'] & & B_2 \arrow[dl, "u_2'"] \\
 & B_1 * B_2 &
\end{tikzcd}$$

is commutative.

The following lemma is an easy exercise in category theory and the proof is omitted.

(4.1) <u>Lemma</u>: Suppose $f_i: A_i \longrightarrow B_i$, $g_i: B_i \longrightarrow C_i$, and $h_i: B_i \longrightarrow D$, $i=1,2$ are morphisms in a category $\mathcal{C}$. Then

1. $(g_1 * g_2)(f_1 * f_2) = g_1 f_1 * g_2 f_2$.

2. $\langle h_1, h_2 \rangle (f_1 * f_2) = \langle h_1 f_1, h_2 f_2 \rangle$.

If (C_1, a_1, m_1) and (C_2, a_2, m_2) are corings in a category $\mathcal{C}$, then a morphism $f: C_1 \longrightarrow C_2$ is called a <u>coring homomorphism</u> if the diagram

$$\begin{tikzcd} C_1 \arrow[r, "f"] \arrow[d, "x_1"] & C_2 \arrow[d, "x_2"] \\ C_1 * C_1 \arrow[r, "f*f"'] & C_2 * C_2 \end{tikzcd}$$

is commutative with $x = a$ and $x = m$. This definition is dual to the notion of ring homomorphism if the ring operations $+$, $\cdot$ are considered as functions from the product of the ring with itself to the ring.

If $\mathcal{C}$ is a category, then we shall denote by $CR(\mathcal{C})$ the category whose objects are corings in $\mathcal{C}$ and whose morphisms are coring homomorphisms. That $CR(\mathcal{C})$ is actually a category is easily verified with the help of Lemma (4.1). We will be concerned in particular with $CR(\mathcal{R})$, the category of corings in the category of rings.

Let $LA(\mathcal{R})$ denote the category whose objects are functors from $\mathcal{R}$ to $\mathcal{R}$ which have left adjoints and whose morphisms are natural transformations of functors. (Actually,

we don't know yet that $LA(\mathfrak{R})$ is a category since we don't know that $[F,G]$ is a set. This will be made clear in the proof of the following theorem.)

(4.2) <u>Theorem</u>: $CR(\mathfrak{R})$ is equivalent to the dual category of $LA(\mathfrak{R})$.

<u>Proof</u>: Define a functor $G:CR(\mathfrak{R})\longrightarrow LA(\mathfrak{R})$ by $G(C,a,m) = [C,-]_{a,m}$; and if $\xi:(C_1,a_1,m_1)\longrightarrow(C_2,a_2,m_2)$ is a coring homomorphism, then $G(\xi):[C_2,-]_{a_2,m_2}\longrightarrow[C_1,-]_{a_1,m_1}$ is the natural transformation $[\xi,-]$. (Recall that if $f:C_2\longrightarrow R$, then $[\xi,R](f) = f\xi$.) We will show that G is a contravariant equivalence.

We know from Theorem (2.4) that $G(C,a,m)$ is an object of $LA(R)$. We must now show that $G(\xi)$ is a natural transformation.

First we must show that $[\xi,R]$ as defined above is a ring homomorphism. If $f,g \in [C_2,R]_{a_2,m_2}$, then

$$[\xi,R](f + g) = [\xi,R](\langle f,g\rangle a_2) = \langle f,g\rangle a_2\xi =$$

$$= \langle f,g\rangle(\xi*\xi)a_1 = \langle f\xi,g\xi\rangle a_1 = \langle[\xi,R](f),[\xi,R](g)\rangle a_1$$

$$= [\xi,R](f) + [\xi,R](g).$$

A similar argument holds for multiplication.

We now show that $[\xi,-]$ is a natural transformation. We must show that if $\alpha:R\longrightarrow S$, then the diagram

$$\begin{array}{ccc} [C_2,R]_{a_2,m_2} & \xrightarrow{[\Phi,R]} & [C_1,R]_{a_1,m_1} \\ \Big\downarrow [C_2,\alpha] & & \Big\downarrow [C_1,\alpha] \\ [C_2,S]_{a_2,m_2} & \xrightarrow{[\Phi,S]} & [C_1,S]_{a_1,m_1} \end{array}$$

is commutative. If $f \in [C_2,R]$, then

$$[C_1,\alpha]\,[\Phi,R]\,(f) = \alpha f \Phi = [\Phi,S]\,[C_2,\alpha]\,(f).$$

Suppose $\Phi:(C_1,a_1,m_1) \longrightarrow (C_2,a_2,m_2)$ and $\Psi:(C_2,a_2,m_2) \longrightarrow (C_3,a_3,m_3)$ are coring homomorphisms. For G to be a contravariant functor, we must show that $G(\Psi\Phi) = G(\Phi)G(\Psi)$, or that $[\Psi\Phi,-] = [\Phi,-]\,[\Psi,-]$. But if $\alpha \in [C_1,R]$, then $[\Psi\Phi,R](\alpha) = \alpha\Psi\Phi = [\Phi,R]\,[\Psi,R]\,(\alpha)$.

We have shown that G is a contravariant functor. We would like to show that G is an equivalence--that G is faithful, full, and representative.

Faithful: Suppose $\Phi,\Psi:(C_1,a_1,m_1) \longrightarrow (C_2,a_2,m_2)$ and $G(\Phi) = G(\Psi)$. Then $[\Phi,-] = [\Psi,-]$, and $[\Phi,C_2](1_{C_2}) = [\Psi,C_2](1_{C_2})$. Then $1_{C_2}\Phi = 1_{C_2}\Psi$, and $\Phi = \Psi$.

Full: Suppose $\eta:[C_2,-] \longrightarrow [C_1,-]$ is a natural transformation. Define $\Phi = \eta_{C_2}(1_{C_2})$. If R is any ring, and

$f:C_2 \longrightarrow R$ is a morphism in $\mathcal{R}$, then from the diagram

$$\begin{tikzcd} {[C_2,C_2]} \arrow[r, "\eta_{C_2}"] \arrow[d, "{[C_2,f]}"] & {[C_1,C_2]} \arrow[d, "{[C_1,f]}"] \\ {[C_2,R]} \arrow[r, "\eta_R"] & {[C_1,R]} \end{tikzcd}$$

we see that

$$\eta_R(f) = \eta_R[C_2,f](1_{C_2}) = [C_1,f]\eta_{C_2}(1_{C_2}) = f\Phi. \qquad *$$

If Φ is a coring homomorphism, then $\eta_R(f) = f\Phi = [\Phi,R](f) = G(\Phi)(f)$, and G will be full. Note that from $*$, $\eta_{C_2*C_2}(u_i) = u_i\Phi$, $i=1,2$, and that $\eta_{C_2*C_2}(a_2) = a_2\Phi$.

$$\begin{aligned} a_2\Phi &= \eta_{C_2*C_2}(a_2) = \eta_{C_2*C_2}(\langle u_1,u_2\rangle a_2) = \eta_{C_2*C_2}(u_1 + u_2) \\ &= \eta_{C_2*C_2}(u_1) + \eta_{C_2*C_2}(u_2) = \langle u_1\Phi,u_2\Phi\rangle a_1 \\ &= \langle u_1,u_2\rangle(\Phi*\Phi)a_1 = (\Phi*\Phi)a_1. \end{aligned}$$

Thus,

$$\begin{tikzcd} C_1 \arrow[r, "\Phi"] \arrow[d, "a_1"] & C_2 \arrow[d, "a_2"] \\ C_1*C_1 \arrow[r, "\Phi*\Phi"] & C_2*C_2 \end{tikzcd}$$

is commutative, and a similar argument holds for comultiplication. Thus, Φ is a coring homomorphism and G is full.

That G is representative follows from Theorem (2.4). Thus, G is a contravariant equivalence.

The remainder of this chapter will be devoted to investigating some properties of the category of corings.

(4.3) <u>Proposition</u>: If the category $\mathcal{C}$ is cocomplete, then so is $CR(\mathcal{C})$, the category of corings in $\mathcal{C}$.

<u>Proof</u>: It is sufficient to show that $CR(\mathcal{C})$ has coproducts and coequalizers.

<u>$CR(\mathcal{C})$ has coproducts</u>: Let (C_i, a_i, m_i), $i \in I$, be a family of corings in $\mathcal{C}$. Since $\mathcal{C}$ is cocomplete, $\coprod_{i\in I} C_i$ exists. Let $C = \coprod_{i\in I} C_i$. Let $a: C \longrightarrow C*C$ be the unique morphism $<(u_i * u_i)a_i>$ which makes the following diagram commutative for each i:

$$\begin{tikzcd}
C_i \arrow[r, "u_i"] \arrow[d, "a_i"'] & C \arrow[d, "a"] \\
C_i * C_i \arrow[r, "u_i * u_i"'] & C*C
\end{tikzcd}$$

Define $m: C \longrightarrow C*C$ in a similar manner.

We claim (C,a,m) is a coring. Let A be any object in $\mathcal{C}$. Then $[C_i,A]_{a_i,m_i}$ is a ring for each $i \in I$. Since the category of rings has products, let $P = \prod_{i \in I} [C_i,A]_{a_i,m_i}$. We claim $P \cong [C,A]_{a,m}$. Define a function $\Phi: P \longrightarrow [C,A]_{a,m}$ in the following manner: Let (f_i) be an element of P where each $f_i: C_i \longrightarrow A$. These f_i induce a map $<f_i>: C \longrightarrow A$ such that $<f_i>u_i = f_i$ for each i. Let $\Phi((f_i)) = <f_i>$. Φ is one-to-one, for if $\Phi((f_i)) = \Phi((g_i))$, then $<f_i> = <g_i>$, and for each i, $<f_i>u_i$ $<g_i>u_i$ and $f_i = g_i$. Thus $(f_i) = (g_i)$. Φ is also onto, for if $h: C \longrightarrow A$, then $hu_i: C_i \longrightarrow A$ for each i, and (hu_i) is an element of P such that $\Phi((hu_i)) = h$. Φ also preserves addition:

$$\Phi((f_i) + (g_i))u_i = \Phi((f_i + g_i))u_i = <f_i + g_i>u_i$$
$$= f_i + g_i = <f_i, g_i>a_i .$$
$$[\Phi((f_i)) + \Phi((g_i))]u_i = (<f_i> + <g_i>)u_i$$
$$= <<f_i>,<g_i>>au_i = <<f_i>,<g_i>>(u_i * u_i)a_i$$
$$= <<f_i>u_i,<g_i>u_i>a_i = <f_i,g_i>a_i .$$

Since these are equal for each i, $\Phi((f_i) + (g_i)) = \Phi((f_i)) + \Phi((g_i))$. A similar argument holds for multiplication and Φ is an isomorphism. Since $[C,A]_{a,m}$ is isomorphic to a ring, it must itself be a ring for any A, and (C,a,m) is a coring.

It must still be shown that (C,a,m) is the coproduct of the (C_i,a_i,m_i) in $CR(\mathcal{C})$. By the way a and m are defined, it is clear that each u_i is a coring homomorphism. Now suppose $f_i:(C_i,a_i,m_i)\longrightarrow(D,a',m')$ is a coring homomorphism for each i. Since C is the coproduct of the C_i in $\mathcal{C}$, there is a unique map $\langle f_i\rangle:C\longrightarrow D$ such that $\langle f_i\rangle u_i = f_i$ for each i. But $\langle f_i\rangle$ is also a coring homomorphism since for each i,

$$a'\langle f_i\rangle u_i = a'f_i = (f_i*f_i)a_i.$$

$$(\langle f_i\rangle*\langle f_i\rangle)au_i = (\langle f_i\rangle*\langle f_i\rangle)(u_i*u_i)a_i$$

$$= (\langle f_i\rangle u_i*\langle f_i\rangle u_i)a_i = (f_i*f_i)a_i.$$

Thus, $a'\langle f_i\rangle = (\langle f_i\rangle*\langle f_i\rangle)a$, and

$$\begin{tikzcd}
C \arrow[r, "\langle f_i\rangle"] \arrow[d, "a"] & D \arrow[d, "a'"] \\
C*C \arrow[r, "\langle f_i\rangle*\langle f_i\rangle"] & D*D
\end{tikzcd}$$

is commutative. Thus $\langle f_i\rangle$ is a coring homomorphism, and (C,a,m) is the coproduct of the (C_i,a_i,m_i) in $CR(\mathcal{C})$.

<u>$CR(\mathcal{C})$ has coequalizers</u>: Suppose $f,g:A\longrightarrow B$ are coring homomorphisms where (A,a_1,m_1) and (B,a_2,m_2) are corings in $\mathcal{C}$. Let $c:B\longrightarrow C$ be the coequalizer of f and g in $\mathcal{C}$. In the diagram

$$\begin{tikzcd}
 & & B*B \arrow[drr, "\alpha*\alpha"] & & \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[ur, "a_2"] \arrow[rr, "\alpha"'] & & C \arrow[r, dashed, "a"'] & C*C
\end{tikzcd}$$

we see that

$$(\alpha*\alpha)a_2f = (\alpha*\alpha)(f*f)a_1 = (\alpha f*\alpha f)a_1 = (\alpha g*\alpha g)a_1 =$$
$$= (\alpha*\alpha)(g*g)a_1 = (\alpha*\alpha)a_2g.$$

Thus, there is a unique morphism $a:C \longrightarrow C*C$ making the above diagram commute. $m:C \longrightarrow C*C$ is defined similarly.

We will show that (C,a,m) is a coring. Let X be any object in $\mathcal{C}$. It is easily verified that $[f,X],[g,X]:[B,X] \longrightarrow [A,X]$ is a pair of ring homomorphisms. Since the category of rings has equalizers, this pair of morphisms will have an equalizer E. E can be thought of as $\{h \in [B,X] \mid hf = hg\}$. We claim $E \cong [C,X]_{a,m}$. Define $\Phi:E \longrightarrow [C,X]$ as follows: If $h \in E$, then $hf = hg$, so there exists $h':C \longrightarrow X$ making the following diagram commute:

$$\begin{tikzcd}
 & & X \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[ur, "h"] \arrow[r, "\alpha"'] & C \arrow[u, "h'"']
\end{tikzcd}$$

Define $\Phi(h) = h'$. If $\Phi(h) = \Phi(k)$, then $h'\alpha = k'\alpha$, and $h = k$. So Φ is one-to-one. If $\gamma:C \longrightarrow X$, then $\gamma\alpha:B \longrightarrow X$, and $\gamma\alpha f = \gamma\alpha g$. Then $\gamma\alpha \in E$ and $\Phi(\gamma\alpha) = \gamma$. Thus, Φ is onto. Φ is additive, for if $h,k \in E$, then

$$[\xi(h + k)]\alpha = \xi(\langle h,k\rangle a_2)\alpha = \langle h,k\rangle a_2.$$

$$[\xi(h) + \xi(k)]\alpha = \langle \xi(h),\xi(k)\rangle a\alpha = \langle \xi(h),\xi(k)\rangle(\alpha*\alpha)a_2$$

$$= \langle \xi(h)\alpha,\xi(k)\alpha\rangle a_2 = \langle h,k\rangle a_2.$$

Since α is an epimorphism, $\xi(h + k) = \xi(h) + \xi(k)$. A similar argument holds for multiplication, and ξ must be a homomorphism. Since $[C,X]_{a,m}$ is isomorphic to a ring, it must itself be a ring for any X, and (C,a,m) is a coring.

We must now show that $\alpha:B \longrightarrow C$ is the coequalizer of f and g in $CR(\mathcal{C})$. It is clear from the way a and m were defined that α is a coring homomorphism. Suppose $h:B \longrightarrow D$ is a coring homomorphism where (D,a',m') is a coring in $\mathcal{C}$, and h is such that $hf = hg$. Since $\alpha:B \longrightarrow C$ is the coequalizer of f and g in $\mathcal{C}$, there is a unique morphism $k:C \longrightarrow D$ making

$$\begin{tikzcd}
 & & D \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[ur, "h"] \arrow[r, "\alpha"'] & C \arrow[u, dashed, "k"']
\end{tikzcd}$$

commute. We would like to show that k is a coring homomorphism. Since $a'k\alpha = a'h$, and

$$(k*k)a\alpha = (k*k)(\alpha*\alpha)a_2 = (k\alpha*k\alpha)a_2 = (h*h)a_2 = a'h,$$

and since α is an epimorphism, we see that

$$\begin{tikzcd}
C \arrow[r, "k"] \arrow[d, "a"'] & D \arrow[d, "a'"] \\
C*C \arrow[r, "k*k"] & D*D
\end{tikzcd}$$

is commutative. A similar argument holds for comultiplication, so k is a coring homomorphism.

Corollary: CR($\mathfrak{R}$) is cocomplete.

Not many other properties of CR($\mathfrak{R}$) are known. Robert Davis has shown [7] that if $\mathfrak{R}_1$ is the category of commutative rings with identity, then the forgetful functor from CR($\mathfrak{R}_1$) to $\mathfrak{R}_1$ has a right adjoint. However, it is not known if this result holds for CR($\mathfrak{R}$). The following result shows that if K is a field, then CR($\mathfrak{A}_K$), with the obvious forgetful functor to the category of sets, does not form an algebraic variety (in the sense of Cohn).

(4.4) Proposition: If K is a field, then CR($\mathfrak{A}_K$) cannot have both products and free objects (relative to the "obvious" forgetful functor).

Proof: Let G be the free K-algebra on one generator x, and let $a(x) = u_1(x) + u_2(x)$, and $m(x) = u_1(x)u_2(x)$. (G,a,m) is then a coring in $\mathfrak{A}_K$. Suppose the product of (G,a,m) with itself exists in CR($\mathfrak{A}_K$), and denote it by (H,a_1,m_1). Define $(0,1):G \longrightarrow H$ as the unique coring homomorphism making

$$\begin{tikzcd} & H \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G & & G \\ & G \arrow[ul, "0"] \arrow[ur, "1_G"'] \arrow[uu, dashed, "{(0,1)}"] & \end{tikzcd}$$

commutative, and define $(1,0)$ similarly. $(0,1)(x)$ and $(1,0)(x)$ are nonzero elements of H. Suppose that (F,a_2,m_2) is the free object in $CR(G_K)$ on one generator y. Define $f:F \longrightarrow H$ by $f(y) = (0,1)(x)\cdot(1,0)(x)$.

$$p_1 f(y) = p_1(0,1)(x)p_1(1,0)(x) = 0x = 0.$$

$$p_2 f(y) = p_2(0,1)(x)p_2(1,0)(x) = x0 = 0.$$

Thus, $p_1 f = p_2 f = 0$, so $f = 0$. But then $(0,1)(x)\cdot(1,0)(x) = 0$, and H has zero divisors, a contradiction to the corollary of Proposition (3.5).

CHAPTER V

THE CATEGORY OF SEMIGROUPS

Many of the methods used to study functors on $\mathcal{R}$ which have left adjoints can also be used to study functors on $\mathcal{S}$, the category of semigroups, which have left adjoints. In particular, Freyd's Theorem (2.3) holds in $\mathcal{S}$, and a functor $T:\mathcal{S}\longrightarrow\mathcal{S}$ will have a left adjoint if and only if T is representable as $[C,-]_m$ where (C,m) is a cosemigroup in $\mathcal{S}$.

Not every ring admits a coring structure. The note following Proposition (3.5) shows that Z, for example, cannot be made into a coring in $\mathcal{R}$. However, every semigroup can be made into a cosemigroup by defining the comultiplication to be one of the injections into the coproduct. If S is a semigroup, then $[S,X]_{u_1}$ is the set $[S,X]$ with left trivial multiplication. For some semigroups a "trivial" comultiplication of this type is the only kind possible.

(5.1) <u>Construction of Coproducts in $\mathcal{S}$</u>: Let S_1 and S_2 be semigroups. Suppose $S_1 \cap S_2 = \phi$, and let $S_1 \cup S_2 = \{s_\lambda | \lambda \in \Lambda\}$. An element of $S_1{*}S_2$ is a product of the form $s_{\lambda_1}s_{\lambda_2}\cdots s_{\lambda_n}$ where no two successive s_{λ_i} belong to the same

S_k, $k=1,2$. $s_{\lambda_1}s_{\lambda_2}\cdots s_{\lambda_n} = s_{\mu_1}s_{\mu_2}\cdots s_{\mu_m}$ if and only if $n = m$ and $s_{\lambda_i} = s_{\mu_i}$ for each i. If $x = s_{\lambda_1}s_{\lambda_2}\cdots s_{\lambda_n}$ and $y = s_{\mu_1}s_{\mu_2}\cdots s_{\mu_m}$ are two elements of $S_1 * S_2$, then xy will be defined as follows:

Case 1: If s_{λ_n} and s_{μ_1} belong to different S_k, then $xy = s_{\lambda_1}s_{\lambda_2}\cdots s_{\lambda_n}s_{\mu_1}s_{\mu_2}\cdots s_{\mu_m}$.

Case 2: If s_{λ_n} and s_{μ_1} belong to the same S_k, then $s_{\lambda_n}s_{\mu_1} = s_\nu \in S_k$. Then $xy = s_{\lambda_1}s_{\lambda_2}\cdots s_{\lambda_{n-1}}s_\nu s_{\mu_2}s_{\mu_3}\cdots s_{\mu_m}$.

For a proof that this construction actually yields the coproduct of S_1 and S_2, the reader is referred to [8].

A semigroup S is said to be <u>periodic</u> if for each $s \in S$ there are integers n and m, $n < m$, such that $s^n = s^m$. In particular, every finite semigroup is periodic.

(5.2) <u>Proposition</u>: Let S be a periodic semigroup. Then (S,m) is a cosemigroup in $\mathcal{S}$ if and only if $m = u_1\Phi$ or $u_2\Phi$ where Φ is an endomorphism of S such that $\Phi^2 = \Phi$.

<u>Proof</u>: Suppose $m = u_1\Phi$ or $u_2\Phi$ where $\Phi^2 = \Phi$. If X is a semigroup and $f,g \in [S,X]$, then $f\cdot g = \langle f,g\rangle u_1\Phi = f\Phi$. $(f\cdot g)\cdot h = (f\Phi)\cdot h = f\Phi^2$. $f\cdot(g\cdot h) = f\Phi$, so the operation is associative and (S,m) is a cosemigroup.

Conversely, suppose (S,m) is a cosemigroup. If $x \in S$, then $m(x) \in S*S$, so $m(x) = s_1 s_2 \ldots s_k$, a finite sequence of elements of $S*S$, where each $s_i \in u_1(S)$ or $u_2(S)$ and no two successive s_i are from the same $u_j(S)$, $j=1,2$.

Case 1. If the length k of $m(x)$ is even, then the length of $m(x^n)$ is nk. But since S is periodic, $x^p = x^n$ for $p < n$. But then $nk = pk$, an impossibility, so the length of $m(x)$ cannot be even.

Case 2: If the length k of $m(x)$ is odd, then the length of $m(x^n)$ is $nk - n + 1$. But if $x^p = x^n$, $p < n$, then $nk - n + 1 = pk - p + 1$; $n(k - 1) = p\ (k - 1)$; and $k = 1$. Thus, the length of $m(x) = 1$, and $m(x) = u_1(x')$ or $m(x) = u_2(x')$ for some x' in S.

If there exist x,y in S such that $m(x) = u_1(x')$ and $m(y) = u_2(y')$, then $m(xy) = u_1(x')u_2(y')$. But this is impossible since we showed above that the length of $m(s)$ is 1 for all s in S. Thus, if $m(x) = u_i(x')$ for some x,x' in S, then for every y in S there is a y' such that $m(y) = u_i(y')$. If we define $\Phi : S \longrightarrow S$ by $\Phi(y) = y'$ where $m(y) = u_i(y')$, then Φ is a homomorphism and it is clear that $m = u_i\Phi$. Since m is associative, $[S,S]_m$ must be associative, so $(1\cdot 1)\cdot 1 = 1\cdot(1\cdot 1)$. $(1\cdot 1)\cdot 1 = \langle\langle 1,1\rangle m, 1\rangle m = \langle \Phi, 1\rangle m = \Phi^2$.

$1\cdot(1\cdot 1) = 1\Phi = \Phi$; so $\Phi^2 = \Phi$.

The remainder of this chapter is devoted to a proof of the fact that $\mathcal{S}$ has exactly two auto-equivalences, I and I_{op}, the identity and opposite functors. Two lemmas are needed, the first of which is apparently due to P. A. Grillet.

We shall define a semigroup P to be <u>projective</u> if, whenever $f:P \longrightarrow Q$ and $g:T \longrightarrow Q$ are semigroup homomorphisms with g <u>onto</u>, then there is a homomorphism $h:P \longrightarrow T$ such that the following diagram is commutative:

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "f"] \\ T \arrow[r, "g"] & Q \end{tikzcd}$$

(5.3) <u>Lemma</u>: Every projective in $\mathcal{S}$ is free.

<u>Proof</u>: Let P be projective and let X be the set of elements in P which are not in P^2. Let S_X be the free semigroup on X. There is a $\Phi:S_X \longrightarrow P$ defined by $\Phi(x) = x$ for all x in X. To show that Φ is onto, note first that there is at least one free semigroup, say S, which maps onto P, say $g:S \longrightarrow P$. Since P is projective, we have $h:P \longrightarrow S$ such that

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "1"] \\ S \arrow[r, "g"] & P \end{tikzcd}$$

is commutative. It follows that h is one-to-one and so P is isomorphic to a subsemigroup of a free semigroup. Then every element of P can be factored into a product of un-factorable elements. Hence, $\Phi: S_X \longrightarrow P$ is onto. And since P is projective, there is Φ' such that

$$\begin{tikzcd} & P \arrow[dl, "\Phi'"'] \arrow[d, "1"] \\ S_X \arrow[r, "\Phi"] & P \end{tikzcd}$$

is commutative. Thus $\Phi\Phi' = 1$. Hence Φ' is one-to-one. To show Φ' is onto it suffices to prove that $\Phi'(x) = x$ for all $x \in X$. Let $\Phi'(x) = x_1 x_2 \ldots x_n$; then $x = \Phi\Phi'(x) =$ $\Phi(x_1)\Phi(x_2)\ldots\Phi(x_n) = x_1 x_2 \ldots x_n$. This shows that $n = 1$, so $x = x_1 = \Phi'(x)$. Hence $\Phi': P \longrightarrow S_X$ is an isomorphism.

If $\langle x \rangle$ is the free semigroup on one generator, then $\langle x \rangle * \langle x \rangle$ can be realized as the free semigroup on two generators, y and z, with injections $u_1(x) = y$; $u_2(x) = z$.

(5.4) Lemma: If $(\langle x \rangle, m)$ is a cosemigroup in $\mathcal{S}$, then m must have one of the following four forms:

(1) $m(x) = yz$

(2) $m(x) = zy$

(3) $m(x) = y$

(4) $m(x) = z$.

<u>Proof</u>: That these four forms yield associative comultiplications is easily verified. In fact, the four forms produce the following functors:

(1) I, the identity functor

(2) I_{op}, the opposite functor

(3) T_l, the left trivial multiplication functor (i.e., it takes a semigroup to the same set but with left trivial multiplication.)

(4) T_r, the right trivial multiplication functor.

To prove the converse, let $m(x)$ be an element of the free semigroup on y and z. Then $m(x) = w(y,z)$ where $w(y,z)$ is some "word" or monomial in y and z. If X is any semigroup and $f,g:\langle x\rangle \longrightarrow X$, then $(f\cdot g)(x) = \langle f,g\rangle m(x) = w(f(x),g(x))$. Since x can be mapped uniquely to any element of X, associativity of $[\langle x\rangle,X]_m$ is equivalent to $w(w(a,b),c) = w(a,w(b,c))$. This will be true for all semigroups only if

$$w(x,w(y,z)) = w(w(x,y),z) \qquad *$$

in the free semigroup on x,y,z. If the number of y's appearing in $w(y,z)$ is n, then the number of x's appearing in $w(x,w(y,z))$ is also n, since each y is replaced by one x. In $w(w(x,y),z)$, however, each y in $w(y,z)$ is replaced by $w(x,y)$, so the number of x's appearing in $w(w(x,y),z)$ is n^n. $*$ implies $n^n = n$, so $n = 0$ or $n = 1$. Thus, $w(y,z)$ contains either one y or no y's. The same holds for z, so we see $m(x)$ must have one of the four listed forms.

The following theorem and the proof thereof were suggested by a similar result obtained by Clark in [9] for the category of rings.

(5.5) Theorem: If $T:\mathcal{S}\longrightarrow\mathcal{S}$ is an equivalence, then either $T \cong I$, or $T \cong I_{op}$, the identity and opposite functors, respectively.

Proof: An equivalence has a left adjoint. Theorem (2.4) and its corollary hold in $\mathcal{S}$ as well as $\mathcal{R}$, so $T \cong [C,-]_m$ where (C,m) is a cosemigroup in $\mathcal{S}$. We see that C must be projective. Suppose $f:A\longrightarrow B$ is onto, and $g:C\longrightarrow B$. Since an equivalence takes extremal epimorphisms to extremal epimorphisms, $[C,f]:[C,A]\longrightarrow[C,B]$ is also onto. Since $g \in [C,B]$, there is $h \in [C,A]$ such that $[C,f](h) = g$. Since $[C,f](h) = fh = g$, C is projective.

Since projectives in $\mathcal{S}$ are free, C must be free on some set X. If $U:\mathcal{S}\longrightarrow Set$ is the forgetful functor, then $U[C,-]_m \cong \prod_X (-)$. Since $[C,-]$ is representative, every semigroup A is isomorphic to $[C,A']$ for some A'. Then the underlying set of A must be the product of X copies of A', so the order of A must either be infinite or $n^{card(X)}$ for some n. Since any set can be made into a semigroup, the above statement can be true for all sets A if and only if card $(X) = 1$. So C is free on one generator and $C = \langle x\rangle$. From Lemma (5.4), T must then be one of the functors I, I_{op},

T_l, or T_r. That I and I_{op} are equivalences is readily verified. T_l and T_r cannot be equivalences, for, if so, they would be representative, and every semigroup would have to be isomorphic to a semigroup with left trivial or right trivial multiplication, which we know is not the case. Thus, the only automorphisms on $\mathcal{S}$ are I and I_{op}.

A property of semigroups is said to be categorical if whenever S has the property, T(S) will also have the property for every auto-equivalence T of $\mathcal{S}$. The above theorem shows that the only semigroup properties which are not categorical are those which are not preserved under the functor I_{op}--those properties which are not left-right symmetric. For example, the property of having a left zero or a left identity would not be categorical.

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BIOGRAPHICAL SKETCH

Burrow Penn Brooks, Jr., was born in Starkville, Mississippi, on May 4, 1943. He graduated from Starkville High School in 1961 and received the degree of Bachelor of Arts with a major in mathematics from Mississippi State University in 1964. He received the degree of Master of Science from Case Institute of Technology in 1966. Since September, 1966, he has attended the University of Florida.

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